2.8 The Derivative as a Function

In this section we let the number $\boldsymbol{a}$ vary. If we replace $\boldsymbol{a}$ in the derivative equation we get:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad\Rightarrow\quad \boxed{f'(x) = \lim_{h\to 0}\frac{f(x-h)-f(x)}{h} = \text{Definition of Derivative}}$$

Remember that $f'(x)$ is called the derivative of f and $f'(x)$ can be interpreted as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Example: a) If $f(x) = 2x^2 - x$, $find\ f'(x)$
b) Compare the graphs of f and f' on the same plot.

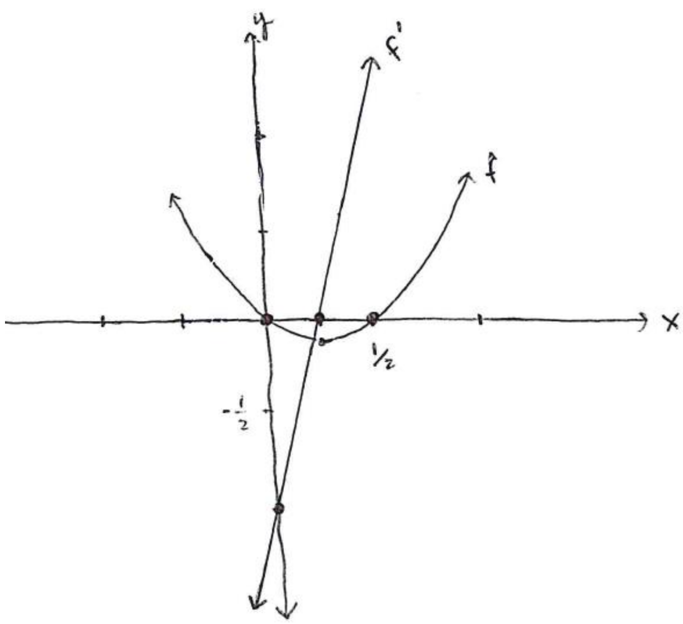
a) $f'(x) = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\frac{[2(x+h)^2-(x+h)]-[2x^2-x]}{h} = \lim\limits_{h\to 0}\frac{[2(x^2+2xh+h^2)-x-h]-[2x^2-x]}{h}$

$= \lim\limits_{h\to 0}\frac{2x^2+2xh+2h^2-x-h-2x^2+x}{h} = \lim\limits_{h\to 0}\frac{4xh+2h^2-h}{h} = \lim\limits_{h\to 0}\frac{h(4x+2h-1)}{h}$

$= \lim\limits_{h\to 0} 4x + 2h - 1$

$\boldsymbol{f'(x) = 4x - 1}$

b) Plot $f(x)$ and $f'(x)$.



Notice that when f '(x) = 0, f(x) has a horizontal tangent line.

- When f '(x) is positive, the tangent line of f(x) has a positive slope.
- When f '(x) is negative, the tangent line of f(x) has a negative slope.
- When f '(x) = 0, the tangent line is horizontal.

Example: a) If $f(x) = \frac{1-2x}{3+x}$, $find\ f'(x)$ b) *State the domain of* $f(x)$ *and* $f'(x)$.

a) $f'(x) = \lim\limits_{h\to 0}\frac{\frac{1-2(x+h)}{3+(x+h)}-\frac{1-2x}{3+x}}{h}$ (find a common denominator for the fraction in the numerator)

$= \lim\limits_{h\to 0}\frac{[1-2(x+h)](3+x)-(1-2x)(3+x+h)}{h(3+x+h)(3+x)} = \lim\limits_{h\to 0}\frac{(1-2x-2h)(3+x)-(1-2x)(3+x+h)}{h(3+x+h)(3+x)}$

$$= \lim_{h\to 0} \frac{(3-6x-6h+x-2x^2-2xh)-(3+x+h-6x-2x^2-2xh)}{h(3+x+h)(3+x)}$$

$$= \lim_{h\to 0} \frac{-6h-h}{h(3+x+h)(3+x)} = \lim_{h\to 0} \frac{-7h}{h(3+x+h)(3+x)} = \lim_{h\to 0} \frac{-7}{(3+x+h)(3+x)} = \frac{-7}{(3+x)^2}$$

b) Domain of f(x): $(-\infty,-3) \cup (-3,\infty)$ Domain of f '(x): $(-\infty,-3) \cup (-3,\infty)$

Other Notations for the Derivative:

If we use ***y = f(x)*** to indicate that the independent variable is ***x*** and the dependent variable is ***y***, then some common alternative notations for the derivative are as follows:

$$\boldsymbol{f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)}$$

The symbol $\boldsymbol{D}$ and $\boldsymbol{\frac{dy}{dx}}$ (and all of the other symbols) are called ***differentiation operators*** because they indicate the operation of differentiation, which is the process of calculating a derivative.

We can write the definition of the derivative in Leibniz notation in the form: $\boldsymbol{\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}}$.

If we want to indicate the value of a derivative $\boldsymbol{\frac{dy}{dx}}$ in Leibniz notation at a specific number or point we use the following notation: $\boldsymbol{\frac{dy}{dx}|_{x=a}}$ or $\frac{dy}{dx}]_{x+a}$ which can also be denoted by $f'(a)$.

Definition: A function ***f*** is differentiable at ***a*** if ***f '(a)*** exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty,\infty)$] if it is differentiable at every number in the interval.

Example: Where is the function f(x) = |x| differentiable?

Solution:

If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{(x+h) - x}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$.

Therefore, for $x < 0$,

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h\to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$.

Now we have to investigate what happens when $x = 0$.

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \quad \text{(if it exists)}$$

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and

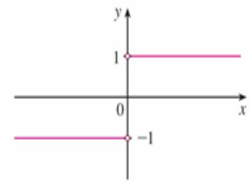
$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b).



$y = f'(x)$

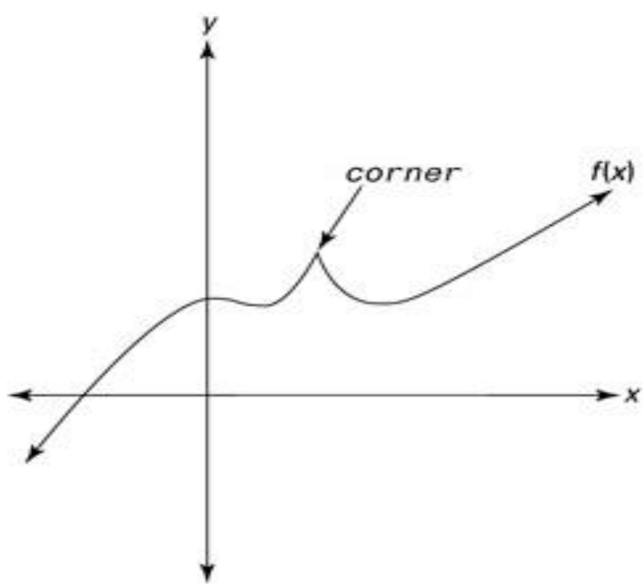
Figure 5(b)

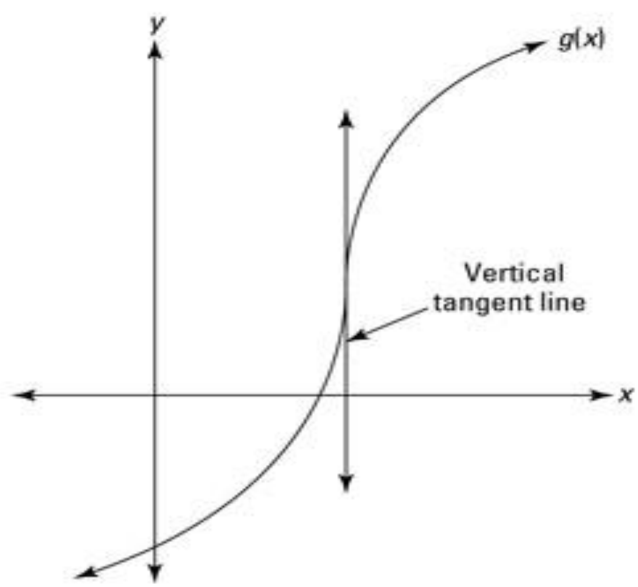
Notice: The **limit** of f(x) = |x| **exists** at x = 0 but the **derivative** of f(x) = |x| **does not exist** at x = 0.

Theorem: If ***f*** is differentiable at ***a***, then ***f*** is continuous at ***a.*** However the converse of this theorem is **not** true. As we see above, f(x) = |x| is continuous at x = 0 but is not differentiable at x = 0.

Here is where the derivatives fail to exist:

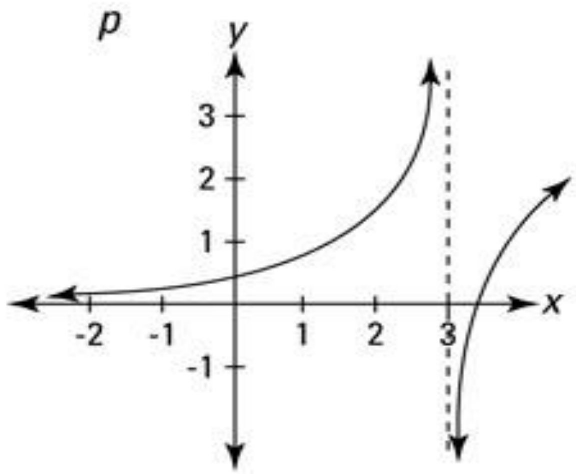
1. Where the function makes are sharp point or corner.
2. Vertical tangents.

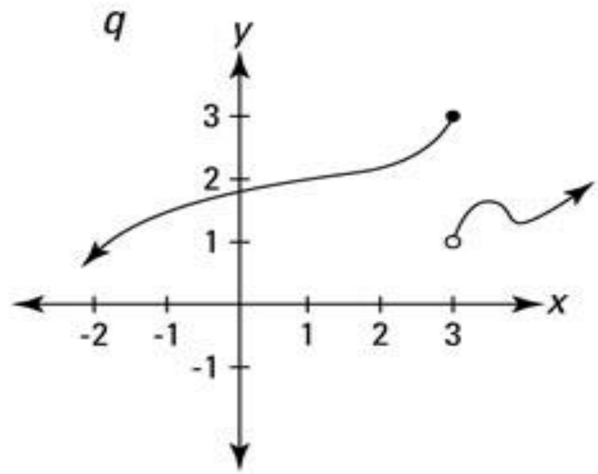




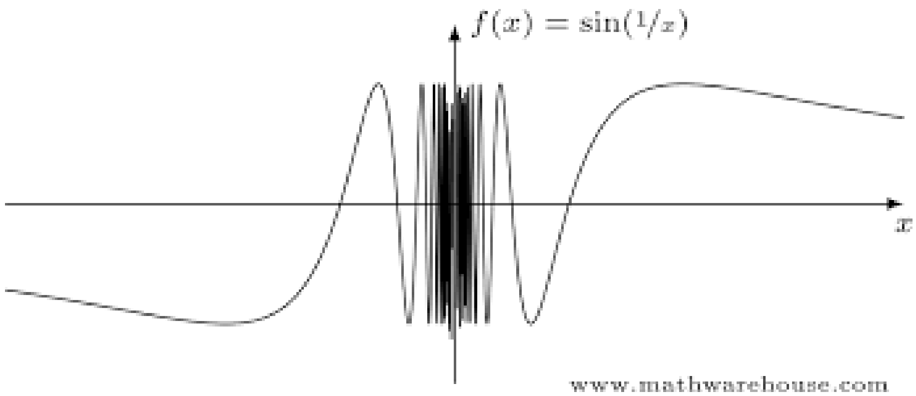
3. Asymptotes

4. Holes, jumps, or gaps or any kind.





4. Oscillations



Higher Derivatives

If f is a differentiable function, then it's derivative, f', is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$.

f'' is called the second derivative of f. Using Leibniz notation, we write the second derivative of **y = f(x)** as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. Other notations include $\boldsymbol{f''(x)}$ *and* $\boldsymbol{y''}$.

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is acceleration. The instantaneous rate of change of velocity with respect to time is called the acceleration ***a(t)***. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$\boldsymbol{a(t) = v'(t) = s''(t)}$ or in Leibniz notation, $\boldsymbol{a = \frac{dv}{dt} = \frac{d^2s}{dt^2}}$

We can also find the third derivative of a function ***f***, which is denoted by ***f‴***. ***f‴*** is the derivative of the second derivative: ***f‴ = (f″)′***. The alternative notation is $\boldsymbol{y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}}$

If the position function is given by ***s(t)***, we can interpret the third derivative by ***s‴ = (s″)′ = a′***

We call the derivative of acceleration function ***jerk***, denoted by ***j***. $\boldsymbol{j = \frac{da}{dt} = \frac{d^3s}{dt^3}}$
Notice that the jerk, ***j***, is the rate of change of acceleration. Since the differentiation process can be continued, we could find the 4th, 5th, 6th, … , nth derivative of a function ***f***, which is denoted by $\boldsymbol{f^{(n)}}$.
We can also write $\boldsymbol{y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}}$.

Example: Using the definition of a derivative and the function $\boldsymbol{f(x) = 3x^2 + 2x + 1}$, find $\boldsymbol{f', f'', \& f'''}$.

$$f' = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{[3(x+h)^2+2(x+h)+1]-[3x^2+2x+1]}{h}$$
$$=\lim_{h\to 0}\frac{[3(x^2+2xh+h^2)+2x+2h+1]-3x^2-2x-1}{h}$$
$$=\lim_{h\to 0}\frac{3x^2+6xh+3h^2+2x+2h+1-3x^2-2x-1}{h}$$
$$=\lim_{h\to 0}\frac{6xh+3h^2+2h}{h}$$
$$=\lim_{h\to 0}\frac{h(6x+3h+2)}{h}$$
$$=\lim_{h\to 0} 6x+3h+2$$
$$\boldsymbol{f'(x) = 6x+2}$$

$$f''(x) = (f')'(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}$$
$$=\lim_{h\to 0}\frac{6(x+h)+2-6x-2}{h}$$
$$=\lim_{h\to 0}\frac{6x+6h+2-6x-2}{h}$$
$$=\lim_{h\to 0}\frac{6h}{h}$$
$$=\lim_{h\to 0} 6$$
$$\boldsymbol{f''(x) = 6}$$

Notice that ***f″(x)*** is a constant function and its graph is the horizontal line ***y = 6***, which means the slope of the tangent will be 0. Therefore for all values of x,

$$\boldsymbol{f'''(x) = 0}.$$